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# A note on the Gauss decomposition of the elliptic Cauchy matrix

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## Abstract

Explicit formulas for the Gauss decomposition of elliptic Cauchy type matrices are derived in a very simple way. The elliptic Cauchy identity is an immediate corollary.

# 1 Introduction

The matrix

$$C = \left( \frac{1}{q_i - r_j} \right)_{i,j=1}^N, \quad q, r \in \mathbb{C}^N, \quad (1)$$

has determinant

$$|C| = \frac{\prod_{1 \leq i < j \leq N} (q_i - q_j)(r_j - r_i)}{\prod_{1 \leq i, j \leq N} (q_i - r_j)}. \quad (2)$$

This identity was first obtained by Cauchy. Now well-known as Cauchy's identity, it has found applications in harmonic analysis, soliton theory, and relativistic Calogero-Moser systems.

Its elliptic generalization involving Weierstrass' sigma function  $\sigma(z)$  is less widely known. It is given by

$$\det \left( \frac{\sigma(q_i - r_j + \lambda)}{\sigma(\lambda)\sigma(q_i - r_j)} \right)_{i,j=1}^N = \frac{\sigma(\lambda + \sum_{k=1}^N (q_k - r_k))}{\sigma(\lambda)} \frac{\prod_{1 \leq i < j \leq N} \sigma(q_i - q_j)\sigma(r_j - r_i)}{\prod_{1 \leq i, j \leq N} \sigma(q_i - r_j)}. \quad (3)$$

This elliptic Cauchy identity dates back to a paper by Frobenius [3]. Like (2), it has shown up in various contexts, giving rise to different proofs, cf. Refs. [1, 6, 5, 4].

Clearly, the identity applies to any minor as well. Moreover, after multiplication from the left and right by diagonal matrices (leading to so-called Cauchy-like matrices) one can still evaluate minors explicitly.

Our perspective, which eventually led to this note, stems from the study of the Lax matrices of Calogero-Moser type systems. In particular, we wished to find the Gauss decomposition of the elliptic Cauchy-like matrix  $C_N(\lambda)$  given by (8) below, i. e., to represent it as

$$C_N(\lambda) = UDL, \quad (4)$$

where  $U$ ,  $D$  and  $L$  are upper-triangular, diagonal and lower-triangular matrices, respectively.

In principle, this decomposition can be obtained by invoking two previously known results. Specifically, the Frobenius formula (3) can be combined with a theorem saying that the elements of the relevant upper- and lower-triangular matrices can be expressed in terms of appropriate minors of the matrix to be decomposed [8]. Indeed, as already mentioned, the minors of an elliptic Cauchy-like matrix also follow from the Frobenius formula.

In this note, we wish to report an alternative method to decompose  $C_N(\lambda)$  which we find interesting and insightful. First of all, it is very economic, inasmuch as an exposition of the proof of the general Gauss decomposition formula and whichever of the known proofs of the Frobenius formula would require far more space and time. Secondly, our direct Gauss decomposition leads to a remarkably simple new proof of the Frobenius formula itself, and also reproduces some other results of interest as easy consequences.

## 2 The decomposition formula

Our proof of the following decomposition formula is self-contained, except for its use of the 3-term identity of the  $\sigma$ -function,

$$\sigma(z+a)\sigma(z-a)\sigma(b+c)\sigma(b-c)+\sigma(z+b)\sigma(z-b)\sigma(c+a)\sigma(c-a)+\sigma(z+c)\sigma(z-c)\sigma(a+b)\sigma(a-b) = 0. \quad (5)$$

We recall that this identity follows directly from the well-known relation between the Weierstrass  $\wp$ -function and the  $\sigma$ -function,

$$\wp(x) - \wp(y) = \frac{\sigma(y+x)\sigma(y-x)}{\sigma(x)^2\sigma(y)^2}, \quad (6)$$

cf. [7]. (Indeed, one need only divide (5) by  $\sigma(z)^2\sigma(a)^2\sigma(b)^2\sigma(c)^2$  and use (6).)

**Theorem.** *Let  $q_1, \dots, q_N, r_1, \dots, r_N, \lambda$  be complex variables and introduce*

$$\lambda_N = \lambda, \quad \lambda_{k-1} := \lambda_k + q_k - r_k \equiv \lambda + \sum_{j=k}^N (q_j - r_j), \quad k = 1, \dots, N. \quad (7)$$

*Define the elliptic Cauchy-like matrix  $C_N(\lambda)$  by*

$$C_N^{ij}(\lambda) := \left( \prod_{k=i+1}^N \frac{\sigma(q_i - r_k)}{\sigma(q_i - q_k)} \right) \frac{\sigma(q_i - r_j + \lambda)}{\sigma(\lambda)\sigma(q_i - r_j)} \left( \prod_{l=j+1}^N \frac{\sigma(q_l - r_j)}{\sigma(r_l - r_j)} \right), \quad i, j = 1, \dots, N, \quad (8)$$

*where is is understood that  $\prod_{k=N+1}^N \dots \equiv 1$ . Then the decomposition (4) is given by*

$$(D^{ii})^{-1} = U^{ii} = L^{ii} = C_i^{ii}(\lambda_i); \quad U^{ij} = C_j^{ij}(\lambda_j), \quad i < j; \quad L^{ij} = C_i^{ij}(\lambda_i), \quad i > j. \quad (9)$$

*Proof.* Substituting

$$2z = q_i - r_j + q_N - r_N, 2a = q_i + r_j - q_N - r_N, 2b = q_i - r_j + q_N - r_N + 2\lambda, 2c = q_i - r_j - q_N + r_N \quad (10)$$

in (5), we obtain an identity from which the first step

$$C_N(\lambda_N) = \begin{pmatrix} I_{N-1} & c_N(\lambda_N) \\ 0 & C_N^{NN}(\lambda_N) \end{pmatrix} \begin{pmatrix} C_{N-1}(\lambda_{N-1}) & 0 \\ 0 & \frac{1}{C_N^{NN}(\lambda_N)} \end{pmatrix} \begin{pmatrix} I_{N-1} & 0 \\ \gamma_N(\lambda_N) & C_N^{NN}(\lambda_N) \end{pmatrix} \quad (11)$$

of an inductive decomposition follows by a straightforward computation. Here,  $I_{N-1}$  stands for the unit  $(N-1) \times (N-1)$  matrix,  $c_N$  is a column vector whose  $(N-1)$  components are  $C_N^{iN}$ ,  $i = 1, \dots, N-1$ , and  $\gamma_N$  is a row vector whose  $(N-1)$  components are  $C_N^{Nj}$ ,  $j = 1, \dots, N-1$ .

Applying the decomposition (11) to the Cauchy matrix  $C_{N-1}(\lambda_{N-1})$  and then to the Cauchy matrix  $C_{N-2}(\lambda_{N-2})$  etc., we arrive directly at the formula (4) with factors (9).  $\square$

It remains to discuss some consequences. First of all, the following result follows effortlessly.

**Corollary (elliptic Cauchy identity).** *The formulas (4) and (9) imply*

$$\det(C_N(\lambda)) = \prod_{k=1}^N C_k^{kk}(\lambda_k) = \prod_{k=1}^N \frac{\sigma(\lambda_{k-1})}{\sigma(\lambda_k)\sigma(q_k - r_k)} = \frac{\sigma(\lambda + \sum_{k=1}^N (q_k - r_k))}{\sigma(\lambda) \prod_{k=1}^N \sigma(q_k - r_k)}, \quad (12)$$

*which is just the identity (3) on account of (8) .*

Secondly, it is worth noting that the decomposition formula provided by the Theorem remains valid if we replace the  $\sigma$ -function by any non-zero odd holomorphic function that satisfies the 3-term identity (5). (Indeed, our proof only uses these properties of  $\sigma(z)$ .) It is known [7] that all such functions are of the form

$$\tilde{\sigma}(z) = e^{\alpha + \beta z^2} \sigma(z), \quad (13)$$

where  $\alpha$  and  $\beta$  are arbitrary complex numbers, and where it is understood that the rational and trigonometric/hyperbolic degenerations of the  $\sigma$ -function are included. If we replace  $\sigma(z)$  by the rational degeneration furnished by  $\tilde{\sigma}(z) = z$  and take  $\lambda$  to infinity, then we obtain the Gauss decomposition of the original Cauchy matrix (1) as well as the determinant formula (2) from our result.

Thirdly, from the trigonometric specialisation we can recover the Gauss decomposition of the Lax matrix of the relativistic trigonometric Calogero-Moser system that recently cropped up in the paper [2] written by two of us. In fact, it was our discussion of the decomposition of the latter matrix that eventually led to the general decomposition encoded in the above Theorem.

Finally, we point out that a similarity transformation with the reversal permutation matrix can be applied to (4) to obtain a ‘lower-diagonal-upper’ version of the decomposition formula. After a relabeling

$$p_1, \dots, p_N \rightarrow p_N, \dots, p_1, \quad p = q, r, \quad (14)$$

this decomposition has a well-defined limit for  $N \rightarrow \infty$ , by contrast to the one in the Theorem.

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